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Extended soliton solutions for the Kaup–Kupershmidt equation

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Abstract

We extend the N-soliton solutions of the Kaup–Kupershmidt equation on a nonzero background decreasing as $(x + 1/a)^{-2}$. These new solutions describe the interaction of N solitary waves with a static bell-shaped wave. We give the conditions so that the Bäcklund transformation relating those solutions and the N-solitons of the Sawada–Kotera equation will be satisfied.

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1. Introduction

The Kaup–Kupershmidt (KK) equation [1]

$$\mathrm{KK}(u) \equiv u_t + \left(u_{xxxx} + \frac{30}{\alpha} u u_{xx} + \frac{45}{2\alpha} u_x^2 + \frac{60}{\alpha^2} u^3\right)_x = 0 \tag{1}$$

is connected to the Sawada–Kotera (SK) equation [2]

$$\mathrm{SK}(u) \equiv u_t + \left(u_{xxxx} + \frac{30}{\alpha} u u_{xx} + \frac{60}{\alpha^2} u^3\right)_x = 0 \tag{2}$$

by the Miura transformation [3]

$$f \tilde{f}_{xx} - 4 f_x \tilde{f}_x + 4 f_{xx} \tilde{f} = 0 \tag{3}$$

with $u_{\mathrm{SK}} = \alpha \partial_x^2 \log f$, $u_{\mathrm{KK}} = \frac{\alpha}{2} \partial_x^2 \log \tilde{f}$.

These two equations are completely integrable: they both possess a third-order Lax pair [1], can be written in bilinear form [4, 5] and are related by reduction to autonomous [6] and nonautonomous [7] Hamiltonian systems of the Hénon–Heiles type. However, there are more difficulties in building the KK solutions than the SK ones.

From the bilinear form of SK, which involves only one Hirota field [8], it is easy to deduce the bilinear Bäcklund transformation (BT) [4] and to reproduce the expression of the N-soliton solution described by a single interaction parameter like other KdV-like equations. Its soliton tau function, obtained by reduction of the BKP hierarchy, can be expressed as a Pfaffian [9, 10].

The bilinear form of KK, involving two Hirota fields, has not yet provided a bilinear BT which can be linearized in the appropriate way and which could lead to a valuable induction process for constructing the N-soliton solutions [11–13]. However, these difficulties have been overcome by establishing the nonlinear superposition formula (NLSF) for KK [14] from its BT [15] obtained by singularity analysis.

The problem that we address in this paper is the construction of a family of solutions, starting from a seed solution of the Lax pair with potential U_0 not identically equal to zero, in order to extend the N-soliton solutions. Thus, we relate the expression of those solutions to a Grammian whose elements are bilinear forms defined on the space of the wavefunctions corresponding to a nonzero potential U_0 and spectral parameters $\{\lambda_i, i = 1, 2, \ldots, N\}$. Therefore, we generalize the expression of the N-soliton tau function obtained in [16] by symmetry reduction of the CKP hierarchy. As an example, we explicitly build the N-parameter solutions of the KK equation for the potential

$$U_0 = \frac{-\alpha}{2}\frac{1}{(x+1/a)^2} \tag{4}$$

and distinguish the two cases $a = 0$ and $a \neq 0$.

We also clarify the link established by the relation (3) between SK and KK in specifying the class of solutions which are involved in this transformation.

The paper is organized as follows. In section 2, the tau function of KK for the extended N-soliton solution is derived from the NLSF. In section 3, the explicit expressions of the extended one- and two-soliton solutions are obtained and their behaviours are graphed for three values of t and two nonzero values of the parameter a. Finally in section 4, we establish the relationship deduced from the BT (3), between the phases of the solitonic solutions of the two dual equations SK and KK.

2. The tau function for KK

The KK equation (1) possesses the Lax pair and the Darboux transformation (DT) [17]:

$$x - \mathrm{Lax}: \left(\partial_x^3 + 6\frac{U}{\alpha}\partial_x + 3\frac{U_x}{\alpha} - \lambda\right)\psi = 0 \tag{5}$$

$$t - \mathrm{Lax}: \left(\partial_t - 9\lambda\partial_x^2 + \left(3\frac{U_{xx}}{\alpha} + 36\frac{U^2}{\alpha^2}\right)\partial_x - 3\frac{U_{xxx}}{\alpha} - 72\frac{UU_x}{\alpha^2} - 36\lambda\frac{U}{\alpha}\right)\psi = 0 \tag{6}$$

$$\mathrm{DT}: u = U + \frac{\alpha}{2}\partial_x^2\log\tau \qquad \tau = \psi\psi_{xx} - \frac{1}{2}\psi_x^2 + 3\frac{U}{\alpha}\psi^2 \qquad \tau_x = \lambda\psi^2 \tag{7}$$

with u and U two solutions of (1).

Its auto-BT is [15]

$$x - \mathrm{BT}: Y_{xx} - \frac{3Y_x^2}{4Y} + 3YY_x + Y^3 + 6\frac{V_x}{\alpha}Y - \lambda = 0 \tag{8}$$

$$\begin{aligned} t - \mathrm{BT}: Y_t - 3Y_{xxxxx} - \frac{3}{2}\Bigg[&60Y^3\left(Y_x + 2\frac{V_x}{\alpha}\right) + 12Y^5 + 10Y\left(Y_{xxx} + 2\frac{V_{xxx}}{\alpha}\right) \\ &+30Y_{xx}\left(Y_x + 2\frac{V_x}{\alpha}\right) + 15Y_x\left(Y_{xx} + 2\frac{V_{xx}}{\alpha}\right) + 30Y^2Y_{xx} \\ &+60Y\left(Y_x + 2\frac{V_x}{\alpha}\right)^2 + 15YY_x^2\Bigg]_x = 0 \end{aligned} \tag{9}$$

$$Y = \frac{v - V}{\alpha} \qquad u = v_x \qquad U = V_x.$$

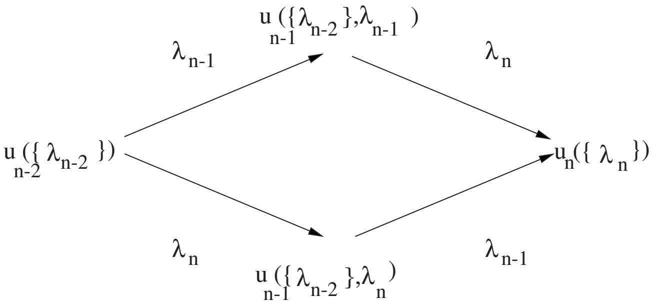

Figure 1. The Bianchi diagram associated with the NLSF.

In [14] we derived the NLSF associated with the Bianchi diagram represented in figure 1: where $(\{\lambda_{n-2}\}, \lambda_{n-1}, \lambda_n)$ corresponds to the set of n particular values $(\lambda_1, \ldots, \lambda_{n-2}, \lambda_{n-1}, \lambda_n)$ of the Bäcklund parameter λ.

Defining the transformation

$$u_k = \frac{\alpha}{2}\partial_x^2 \log f_k \qquad (k = 1, 2, \ldots, n) \tag{10}$$

the NLSF is written as

$$f_n(\{\lambda_n\}) = f_{n-2}(\{\lambda_{n-2}\}) \begin{vmatrix} \frac{f_{n-1}(\{\lambda_{n-2}\},\lambda_{n-1})}{f_{n-2}(\{\lambda_{n-2}\})} & R_n(\{\lambda_{n-2}\}, \lambda_{n-1}, \lambda_n) \\ R_n(\{\lambda_{n-2}\}, \lambda_{n-1}, \lambda_n) & \frac{f_{n-1}(\{\lambda_{n-2}\},\lambda_n)}{f_{n-2}(\{\lambda_{n-2}\})} \end{vmatrix} \tag{11}$$

$$R_n(\{\lambda_{n-2}\}, \lambda_{n-1}, \lambda_n) = \int^x \sqrt{\left(\frac{f_{n-1}(\{\lambda_{n-2}\}, \lambda_{n-1})}{f_{n-2}(\{\lambda_{n-2}\})}\right)_x \left(\frac{f_{n-1}(\{\lambda_{n-2}\}, \lambda_n)}{f_{n-2}(\{\lambda_{n-2}\})}\right)_x}\, \mathrm{d}x. \tag{12}$$

Expression (11) allows us to write explicitly the tau function depending on N distinct parameters $\{\lambda_N\}$ as a Gram determinant:

$$\tau^{(N)} \equiv f_N = f_0 \det\left[\int^x \psi(\lambda_j)\psi(\lambda_l)\,\mathrm{d}x\right]_{1\leqslant j,l\leqslant N} \tag{13}$$

where f_0 is related to the seed solution U_0 by formula (10) and $\{\psi(\lambda_j)\}$ are solution of (5) and (6) with potential U_0 and spectral parameter λ_j.

When $U_0 = 0$ $(f_0 = 1)$ the result (13) has been obtained by symmetry reduction of the CKP hierarchy [16]. Here we extend its validity to a non zero background potential by iterating the formula (11) in the following way. Considering the x-derivative of the ratio f_{p-1}/f_{p-2}:

$$\left(\frac{f_{p-1}(\{\lambda_{p-2}\}, \lambda_a)}{f_{p-2}(\{\lambda_{p-2}\})}\right)_x = \frac{f_{p-1,x}(\{\lambda_{p-2}\}, \lambda_a)}{f_{p-2}(\{\lambda_{p-2}\})} - \frac{f_{p-1}(\{\lambda_{p-2}\}, \lambda_a) f_{p-2,x}(\{\lambda_{p-2}\})}{f_{p-2}^2(\{\lambda_{p-2}\})} \tag{14}$$

we eliminate in the right-hand side $f_{p-1}(\{\lambda_{p-2}\}, \lambda_a)$ and its x derivative in using (11) for $n \equiv p - 1$:

$$\left(\frac{f_{p-1}(\{\lambda_{p-2}\}, \lambda_a)}{f_{p-2}(\{\lambda_{p-2}\})}\right)_x = \left(\frac{f_{p-2}(\{\lambda_{p-3}\}, \lambda_a)}{f_{p-3}(\{\lambda_{p-3}\})} - \frac{f_{p-3}(\{\lambda_{p-3}\})}{f_{p-2}(\{\lambda_{p-3}\}, \lambda_{p-2})} R_{p-1}^2(\{\lambda_{p-3}\}, \lambda_{p-2}, \lambda_a)\right)_x. \tag{15}$$

Taking account of (12) for $n \equiv p - 1$ this last expression can equivalently be written as

$$\left(\frac{f_{p-1}(\{\lambda_{p-2}\}, \lambda_a)}{f_{p-2}(\{\lambda_{p-2}\})}\right)_x = \left(\sqrt{\left(\frac{f_{p-2}(\{\lambda_{p-3}\}, \lambda_a)}{f_{p-3}(\{\lambda_{p-3}\})}\right)_x}\right.$$

$$-\frac{f_{p-3}(\{\lambda_{p-3}\})}{f_{p-2}(\{\lambda_{p-3}\},\lambda_{p-2})}\sqrt{\left(\frac{f_{p-2}(\{\lambda_{p-3}\},\lambda_{p-2})}{f_{p-3}(\{\lambda_{p-3}\})}\right)_x} R_{p-1}(\{\lambda_{p-3}\},\lambda_{p-2},\lambda_a)\Bigg)^2. \tag{16}$$

A similar expression to (16) where λ_a is replaced by λ_b is also considered:

$$\begin{aligned}\left(\frac{f_{p-1}(\{\lambda_{p-2}\},\lambda_b)}{f_{p-2}(\{\lambda_{p-2}\})}\right)_x &= \Bigg(\sqrt{\left(\frac{f_{p-2}(\{\lambda_{p-3}\},\lambda_b)}{f_{p-3}(\{\lambda_{p-3}\})}\right)_x}\\ &\quad -\frac{f_{p-3}(\{\lambda_{p-3}\})}{f_{p-2}(\{\lambda_{p-3}\},\lambda_{p-2})}\sqrt{\left(\frac{f_{p-2}(\{\lambda_{p-3}\},\lambda_{p-2})}{f_{p-3}(\{\lambda_{p-3}\})}\right)_x} R_{p-1}(\{\lambda_{p-3}\},\lambda_{p-2},\lambda_b)\Bigg)^2.\end{aligned} \tag{17}$$

Making the product of the two relations (16) and (17), extracting the square roots of the two members and integrating with respect to x, we eliminate the remaining square roots by using expression (12) for respectively $n \equiv p$ and $p-1$ and obtain the following relation:

$$\begin{aligned}R_p(\{\lambda_{p-2}\},\lambda_a,\lambda_b) &= R_{p-1}(\{\lambda_{p-3}\},\lambda_a,\lambda_b)\\ &\quad -\frac{f_{p-3}(\{\lambda_{p-3}\})}{f_{p-2}(\{\lambda_{p-2}\})}R_{p-1}(\{\lambda_{p-2}\},\lambda_a)R_{p-1}(\{\lambda_{p-2}\},\lambda_b)\end{aligned} \tag{18}$$

which generalizes for arbitrary value of $p > 3$ the expression obtained for $p = 3$ in [14].

With the use of (11) and (18), we can then prove the equality between a $(N-m)\times(N-m)$ determinant and a $(N-m+1)\times(N-m+1)$ determinant:

$$\begin{aligned}&f_m(\{\lambda_m\})\\ &\times\begin{vmatrix}\frac{f_{m+1}(\{\lambda_m\},\lambda_{m+1})}{f_m(\lambda_m)} & R_{m+2}(\{\lambda_m\},\lambda_{m+1},\lambda_{m+2}) & \cdots & R_{m+2}(\{\lambda_m\},\lambda_{m+1},\lambda_N)\\ R_{m+2}(\{\lambda_m\},\lambda_{m+1},\lambda_{m+2}) & \frac{f_{m+1}(\{\lambda_m\},\lambda_{m+2})}{f_m(\{\lambda_m\})} & \cdots & \vdots\\ \vdots & \vdots & \vdots & \vdots\\ R_{m+2}(\{\lambda_m\},\lambda_{m+1},\lambda_N) & R_{m+2}(\{\lambda_m\},\lambda_{m+2},\lambda_N) & \cdots & \frac{f_{m+1}(\{\lambda_m\},\lambda_N)}{f_m(\{\lambda_m\})}\end{vmatrix}\\ &= f_{m-1}(\{\lambda_{m-1}\})\\ &\times\begin{vmatrix}\frac{f_m(\{\lambda_{m-1}\},\lambda_m)}{f_{m-1}(\{\lambda_{m-1}\})} & R_{m+1}(\{\lambda_{m-1}\},\lambda_m,\lambda_{m+1}) & \cdots & R_{m+1}(\{\lambda_{m-1}\},\lambda_m,\lambda_N)\\ R_{m+1}(\{\lambda_{m-1}\},\lambda_m,\lambda_{m+1}) & \frac{f_m(\{\lambda_{m-1}\},\lambda_{m+1})}{f_{m-1}(\{\lambda_{m-1}\})} & \cdots & \vdots\\ \vdots & \vdots & \vdots & \vdots\\ R_{m+1}(\{\lambda_{m-1}\},\lambda_m,\lambda_N) & R_{m+1}(\{\lambda_{m-1}\},\lambda_{m+1},\lambda_N) & \cdots & \frac{f_m(\{\lambda_{m-1}\},\lambda_N)}{f_{m-1}(\{\lambda_{m-1}\})}\end{vmatrix}.\end{aligned} \tag{19}$$

We now consider (11) and make the identification $n \equiv N$. By applying (19) $N-2$ times we obtain

$$F(\{\lambda_N\}) = f_0\begin{vmatrix}\frac{f_1(\lambda_1)}{f_0} & R_2(\lambda_1,\lambda_2) & \cdots & R_2(\lambda_1,\lambda_N)\\ R_2(\lambda_1,\lambda_2) & \frac{f_1(\lambda_2)}{f_0} & \cdots & R_2(\lambda_2,\lambda_N)\\ \vdots & \vdots & \vdots & \vdots\\ R_2(\lambda_1,\lambda_N) & R_2(\lambda_2,\lambda_N) & \cdots & \frac{f_1(\lambda_N)}{f_0}\end{vmatrix} \tag{20}$$

where $f_1(\lambda_j)$, $j = 1,\ldots,N$ is a one-parameter tau function associated with background U_0.

Considering the DT (7) with $U \equiv U_0$ we have that

$$u(\lambda_j) = \frac{\alpha}{2}\partial_x^2\log\left[f_0\int^x\psi^2(\lambda_j)\right] \equiv \frac{\alpha}{2}\partial_x^2\log f_1(\lambda_j) \tag{21}$$

with $\psi(\lambda_j)$ solution of (5) for $U \equiv U_0$ and $\lambda \equiv \lambda_j$.

Therefore

$$\frac{f_1(\lambda_j)}{f_0} = \int^x \psi^2(\lambda_j) \tag{22}$$

which proves that (20) is identical to (13).

3. Extended solitons for KK

We here construct new solutions of KK starting from (4) which corresponds to $f_0 = 1 + ax$. The general solution of (5) and (6) with spectral parameter λ, becomes

$$\begin{aligned}\psi(\lambda) = (1+ax)^{-1}(A\mathrm{e}^{-pax-9a^5p^5t}(1+p+pax) + B\mathrm{e}^{-rax-9a^5r^5t}(1+r+rax)\\ +C\mathrm{e}^{-sax-9a^5s^5t}(1+s+sax))\end{aligned} \tag{23}$$

where $a^3p^3 = a^3r^3 = a^3s^3 = -\lambda$ and A, B, C are arbitrary constants.

With the use of (13), it is now easy to construct extended N-soliton solutions for KK. We only give explicitly the one- and two-parameter solutions.

Setting $C = 0$, the expression (21) of the one-parameter solution becomes

$$\begin{aligned}u(k) = \frac{\alpha}{2}\partial_x^2 \log\Bigg[1 + ax + a\frac{3+\sqrt{3}\mathrm{i}}{k} + 4\left(1 + ax + a\frac{\sqrt{3}\mathrm{i}}{k}\right)\mathrm{e}^{kx-k^5t+\delta}\\ +\left(1 + ax + a\frac{-3+\sqrt{3}\mathrm{i}}{k}\right)\mathrm{e}^{2(kx-k^5t+\delta)}\Bigg]\end{aligned} \tag{24}$$

$$\delta = \frac{Br}{A(p+r)} \tag{25}$$

where $k = a(p-r)$ and $\mathrm{i}^2 = -1$.

Its asymptotic behaviour ($|t| \gg 1$) is

(i) in the reference frame of the soliton $\xi = x - k^4t + \delta/k$:

$$\mathrm{Re}\,(u(k)) \sim \frac{\alpha}{2}\partial_x^2 \log[1 + 4\mathrm{e}^{k\xi} + \mathrm{e}^{2k\xi}] \tag{26}$$

which corresponds to a moving solitary wave with speed $v = k^4$;

(ii) outside the reference frame of the one soliton, for $t \to \pm\infty$,

$$f_1(k) \sim \frac{\sqrt{3}}{k}(\mathrm{i} \pm \sqrt{3}) + \frac{1}{a} + x \tag{27}$$

$$\mathrm{Re}\,(u(k)) \sim \frac{\alpha}{2}\frac{3/k^2 - X^2}{(3/k^2 + X^2)^2} \qquad X = \pm\frac{3}{k} + \frac{1}{a} + x \tag{28}$$

which corresponds to a static bell-shaped wave.

With the two-parameter tau function

$$f_2(\lambda_1, \lambda_2) = f_0 \begin{vmatrix} \int^x \psi^2(\lambda_1) & \int^x \psi(\lambda_1)\psi(\lambda_2) \\ \int^x \psi(\lambda_1)\psi(\lambda_2) & \int^x \psi^2(\lambda_2) \end{vmatrix} \tag{29}$$

the extended two-soliton solution becomes

$$u(k_1, k_2) = \frac{\alpha}{2}\partial_x^2 \log f_2(k_1, k_2) \tag{30}$$

$$f_2(k_1, k_2) = 1 + ax + a\frac{(k_1+k_2)(\sqrt{3}\mathrm{i}+3)}{k_1k_2} + 4\left(1 + ax + a\frac{(\sqrt{3}\mathrm{i}+3)}{k_2} + a\frac{\sqrt{3}\mathrm{i}}{k_1}\right)\mathrm{e}^{\theta_1}$$

$$+4\left(1+ax+a\frac{\sqrt{3}\mathrm{i}+3}{k_1}+a\frac{\sqrt{3}\mathrm{i}}{k_2}\right)\mathrm{e}^{\theta_2}+\left(1+ax+a\frac{\sqrt{3}\mathrm{i}+3}{k_2}+a\frac{\sqrt{3}\mathrm{i}-3}{k_1}\right)\mathrm{e}^{2\theta_1}$$
$$+\left(1+ax+a\frac{\sqrt{3}\mathrm{i}+3}{k_1}+a\frac{\sqrt{3}\mathrm{i}-3}{k_2}\right)\mathrm{e}^{2\theta_2}$$
$$+8B_{12}\left(1+ax+a\frac{\sqrt{3}\mathrm{i}(k_1+k_2)}{k_1k_2}\right)\mathrm{e}^{\theta_1+\theta_2}$$
$$+4A_{12}\left(\left(1+ax+a\frac{\sqrt{3}\mathrm{i}}{k_2}+a\frac{\sqrt{3}\mathrm{i}-3}{k_1}\right)\mathrm{e}^{2\theta_1+\theta_2}\right.$$
$$\left.+\left(1+ax+a\frac{\sqrt{3}\mathrm{i}}{k_1}+a\frac{\sqrt{3}\mathrm{i}-3}{k_2}\right)\mathrm{e}^{\theta_1+2\theta_2}\right)$$
$$+A_{12}^2\left(1+ax+a\frac{(\sqrt{3}\mathrm{i}-3)(k_1+k_2)}{k_1k_2}\right)\mathrm{e}^{2(\theta_1+\theta_2)} \tag{31}$$

$$\theta_j=k_jx-k_j^5t+\tilde{\delta}_j \qquad j=1,2$$
$$\tilde{\delta}_j=\frac{B_j(r_j-p_l)q_j(p_j+p_l)}{A_j(r_j+p_l)(p_j+r_j)(p_j-p_l)} \qquad j,l=1,2 \qquad j\neq l \tag{32}$$
$$A_{jl}=\frac{(k_j-k_l)^2(k_j^2-k_jk_l+k_l^2)}{(k_j+k_l)^2(k_j^2+k_jk_l+k_l^2)} \tag{33}$$
$$B_{jl}=\frac{2k_j^4-k_j^2k_l^2+2k_l^4}{(k_j+k_l)^2(k_j^2+k_jk_l+k_l^2)} \qquad j\neq l. \tag{34}$$

Outside the reference frames of the solitons characterized by k_1 and k_2 we have for $t\pm\infty$:

$$f_2(k_1,k_2)\sim\frac{\sqrt{3}(k_1+k_2)}{k_1k_2}(\mathrm{i}\pm\sqrt{3})+\frac{1}{a}+x \tag{35}$$
$$\mathrm{Re}\,(u(k_1,k_2))\sim\frac{\alpha}{2}\frac{3(k_1+k_2)^2/(k_1^2k_2^2)-X^2}{(3(k_1+k_2)^2/(k_1^2k_2^2)+X^2)^2} \qquad X=\pm\frac{3(k_1+k_2)}{k_1k_2}+\frac{1}{a}+x. \tag{36}$$

Note that setting $a=0$ in (24) and (30) we recover the usual expressions for the one- and two-soliton solutions of KK [14].

In figures 2–7 we draw the behaviour of the extended one- and two-solitons for two values of the parameter a, respectively $a=0.5$ (dotted curve) and $a=500$ (full curve).

4. Miura transformation between SK and KK

With the seed solution $\tilde{f}_0=1+ax$, equation (3) is satisfied either for $f_0=1$ or $(1+ax)^2$. This last expression corresponds to a solution of SK:

$$u_{\mathrm{SK}}=-\frac{2\alpha a^2}{(1+ax)^2} \tag{37}$$

which belongs to the second family of movable poles possessing one negative Fuchs index [15]. This solution cannot be iterated for $a\neq 0$, therefore it is not taken into account.

From the bilinear form of SK it is well known [8] that one can write the N-soliton tau function as

$$f_N=\sum_{\mu_j,\mu_l=0,1}\exp\left(\sum_{j=1}^{N}\mu_j\theta_j+\sum_{1\leqslant j\leqslant l}\mu_j\mu_l\log A_{jl}\right) \tag{38}$$

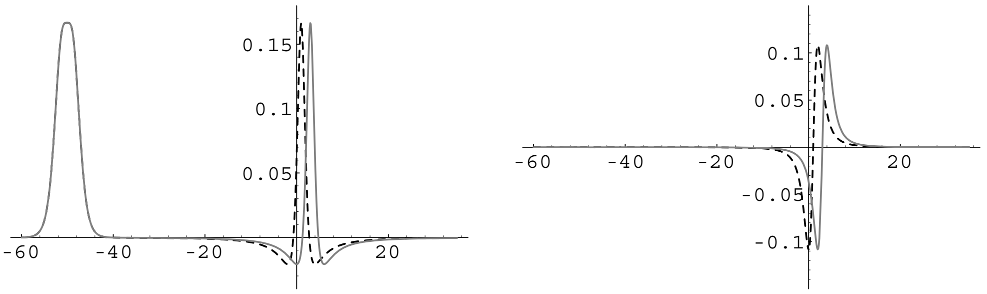

Figure 2. Real and imaginary part of $u(k = 1)$ at $t = -50$.

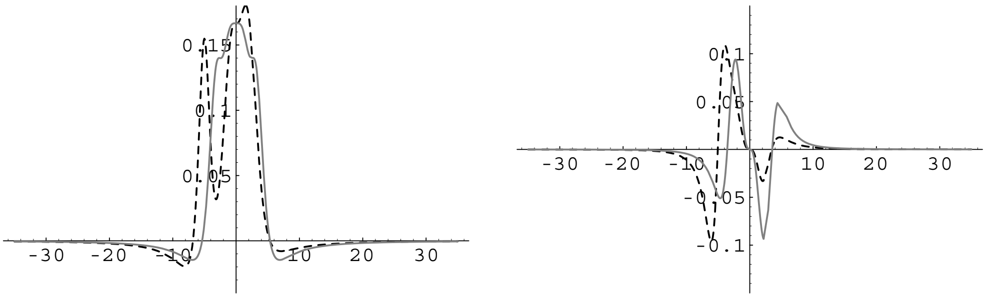

Figure 3. Real part and imaginary part of $u(k = 1)$ at $t = 0$.

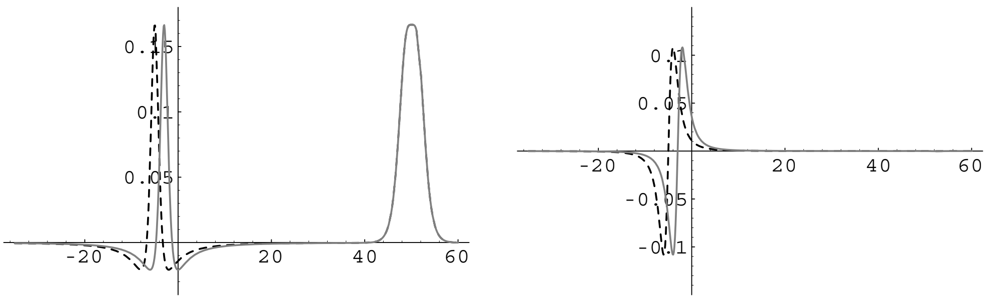

Figure 4. Real part and imaginary part of $u(k = 1)$ at $t = 50$.

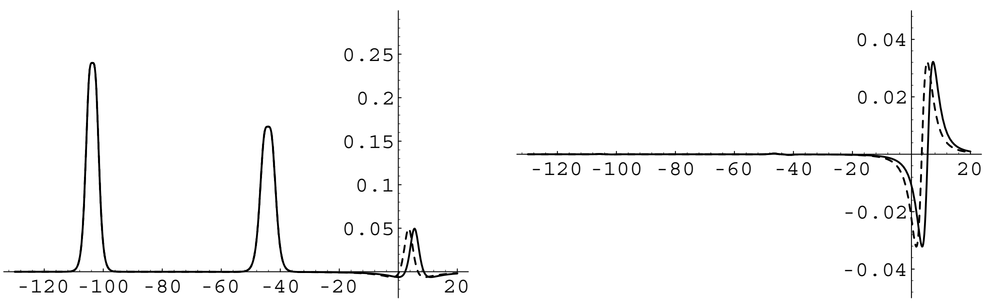

Figure 5. Real part and imaginary part of $u(k_1 = 1, k_2 = 1.2)$ at $t = -50$.

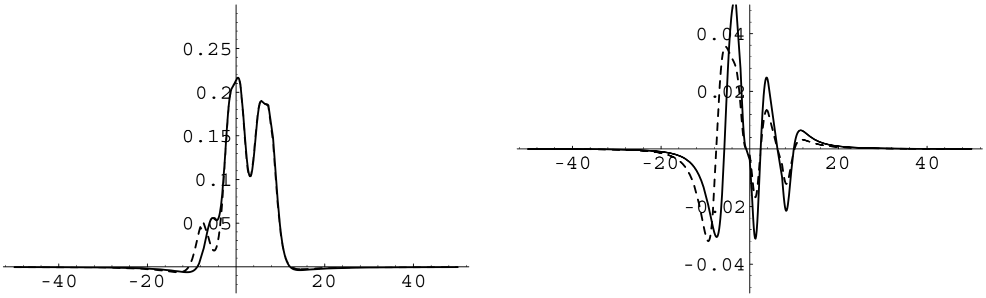

Figure 6. Real and imaginary part of $u(k_1 = 1, k_2 = 1.2)$ at $t = 0$.

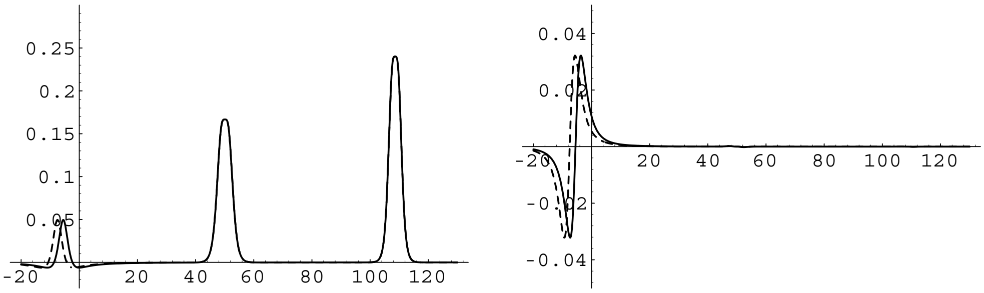

Figure 7. Real and imaginary part of $u(k_1 = 1, k_2 = 1.2)$ at $t = 50$.

where $\theta_j = k_j x - k_j^5 t + \delta_j$ and A_{jl} defined by (33).

To be satisfied, the BT (3) imposes some relationship between the phases of the soliton solutions of SK and KK, such that to a regular (singular) soliton solution of SK corresponds a singular (regular) solution of KK. This also holds for extended soliton solutions of KK.

For example, considering in (3) the two-soliton tau function of SK and KK:

$$f = 1 + \varepsilon_1 e^{\theta_1} + \varepsilon_2 e^{\theta_2} + \varepsilon_1\varepsilon_2 A_{12} e^{\theta_1+\theta_2} \qquad \varepsilon_j = e^{\delta_j} \tag{39}$$

$$\begin{aligned}\tilde{f} = 1 &+ 4(\tilde{\varepsilon}_1 e^{\theta_1} + \tilde{\varepsilon}_2 e^{\theta_2}) + \tilde{\varepsilon}_1^2 e^{2\theta_1} + \tilde{\varepsilon}_2^2 e^{2\theta_2} + \tilde{\varepsilon}_1\tilde{\varepsilon}_2 B_{12} e^{\theta_1+\theta_2} \\ &+ 4A_{12}(\tilde{\varepsilon}_1\tilde{\varepsilon}_2^2 e^{\theta_1+2\theta_2} + \tilde{\varepsilon}_1^2\tilde{\varepsilon}_2 e^{\theta_2+2\theta_1}) + \tilde{\varepsilon}_1^2\tilde{\varepsilon}_2^2 A_{12}^2 e^{2(\theta_1+\theta_2)} \qquad \tilde{\varepsilon}_j = e^{\tilde{\delta}_j}\end{aligned} \tag{40}$$

the following relation between the phases appears:

$$\tilde{\varepsilon_j} = -\varepsilon_j \qquad j = 1, 2 \tag{41}$$

and it remains valid if $\tilde{f}$ is replaced by the extended two-soliton solution (30).

The relation (41) is mainly determined by the coefficient in (3) of e^{θ_j}, $j = 1, 2$, which must be equal to zero. Therefore, it is easy to check that it remains valid for $j = 1, \ldots, N$ with N arbitrary.

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